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NOTE

A NOTE ON BUBBLES, WORTHLESS ASSETS, AND THE CURIOUS CASE OF GENERAL MOTORS

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Since the company declared bankruptcy in June 2009, shares of General Motors stock (now known as Motors Liquidation Company) have continued to trade at a high volume while maintaining a market capitalization near $300 million through most of 2010. Anecdotal evidence strongly suggests that both rational speculators and uninformed investors (often mistaking Motors Liquidation for the new, reorganized GM) have purchased the stock. We develop a theoretical asset-pricing model that includes both types of agents. We present two major results. First, the most frequent state is one where a small fraction of rational agents ensure that the share price behaves as if all agents are rational. A second state exists where all rational agents exit and the share price is inflated. Second, fitting the model to Motors Liquidation, we find evidence of irrational asset pricing for this firm. We find little evidence of similar behavior in the share prices of the thirty stocks that compose the Dow Jones Industrial Average.

Keywords: Asset Pricing, Heterogeneous Agents, General Motors

1. INTRODUCTION

On June 1, 2009, General Motors filed for Chapter 11 bankruptcy. The firm's common stock, then sold as Motors Liquidation Company and traded under the symbol MTLQQ, maintained a market capitalization near $300 million until November 2010, and its daily share volume at times exceeded those of Procter & Gamble, Coca-Cola, Apple, IBM, and Google.

Trading shares of bankrupt firms is neither uncommon nor clear evidence of irrational behavior. Shareholders frequently receive some compensation for their holdings, and the potential for a government bailout of shareholders was especially noteworthy for MTLQQ. The mere fact that MTLQQ was heavily traded is not evidence that its share price and market capitalization were detached from their fundamental values. In the case of MTLQQ, however, the consensus of financial

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observers is that its fundamental value was negligible, and that its inflated share price was the product of a bubble equilibrium. On July 1, 2009, MTLQQ issued a press statement confirming its "strong belief" that stockholders would eventually lose all of their investment. The company's website has repeatedly issued similar warnings. Furthermore, in our research, we are unable to identify a single respected professional analyst who believes that MTLQQ's fundamental value even remotely justified its market capitalization from June 2009 through November 2010.

Trading in MTLQQ has generated considerable interest in the popular press, including the *New York Times*, the *Financial Times*, *CNN*, and *Fortune*.[1] These stories have identified three major groups who are purchasing or holding Motors Liquidation stock: (1) investors who purchase the stock either in a mistaken attempt to purchase shares of the "new GM" (which was not publicly traded until November 2010),[2] or as part of a misguided attempt to "buy low"; (2) investors for whom the decline in share price has made their holdings too small to merit the effort or transaction costs associated with selling the stock; and (3) speculators acting rationally. For the remainder of the paper we refer to all investors other than rational speculators as *uninformed*.[3]

On November 17, 2010, the reorganized General Motors launched an initial public offering and began trading under the symbol GM. The share price of MTLQQ rapidly collapsed, decreasing by 40% in a week, and it has never returned to the neighborhood of its earlier prices. General Motor's IPO, however, had no apparent relationship to MTLQQ's fundamental value. The only notable fundamental shock to MTLQQ after June 2009 occurred in March 2011, when a bankruptcy court approved the firm's liquidation and announced that its shares would be legally worthless as of December 15, 2011. This news, however, had no discernible effect on MTLQQ's share price. These two events support the hypothesis that much of MTLQQ's earlier market capitalization resulted from mistaken attempts to buy shares in the new GM.

This paper develops an asset-pricing model for a stock that is fundamentally worthless. Consistent with the anecdotal evidence, we allow both rational investors and uninformed investors to purchase the stock. Uninformed investors behave mechanically and purchase the stock only at its spot price. They cannot, for example, take short positions. When there are an elevated number of uninformed investors, the share price is bid up. Rational investors, expecting the share price to converge downward to its steady state, exit the market. When there are a decreased number of uninformed investors, however, rational investors enter based on the expectation that as the number of uninformed investors converges back to its steady state, the share price will rebound. Rational agents therefore act as a price floor. Equilibrium switches between a state where the share price behaves as if all agents are rational and another where only uninformed agents hold the stock.

We present two main results. First, calibrating the model to MTLQQ, we find that that the rational state is far more common than the uninformed state. Furthermore, even in the rational state, the vast majority of shares are held by uninformed agents. A relatively small share of rational agents is thus sufficient to make the market as a whole act rationally most of the time. *Fortune's* description of trading in Motors Liquidation as one of the "dumbest moments in business 2009" therefore applies only to some trading days. Our second main result is to provide evidence that MTLQQ's share price was frequently detached from fundamentals between June 2009 and November 2010. Although we cannot directly observe the state of MTLQQ for a given day, our theoretical model predicts that when the share price is above average, it will exhibit more volatility and less persistence than when it is below average. A model where the share price always equals its fundamental value makes no such predictions. Fitting MTLQQ's share price to an AR(1) process confirms the predictions of our theoretical model. We perform a similar analysis for the 30 stocks that compose the Dow Jones Industrial Average and find little evidence of bubble behavior. MTLQQ thus appears unique.

Two related literatures are relevant to the present paper. First, the bounded rationality literature in macroeconomics often allows different sets of agents to simultaneously possess different levels of rationality.[4] Second, the behavioral finance literature often cites psychological factors, such as a tendency to overreact to negative news, as enabling share prices not to equal their fundamental values.[5]

2. MODEL

We model a fundamentally worthless stock with N publicly traded shares. We make two core assumptions. First, in each period, *uninformed investors* devote an exogenous amount of wealth, denoted as τ_t, to purchase the asset. Furthermore, a constant fraction of the shares held by uninformed investors, λ, is sold each period.[6] When an AR(1) distribution is imposed on $\ln(\tau_t)$, and R_t denotes the number of shares held by uninformed investors, these assumptions yield the following:

$$\tau_t = \tau_{t-1}^{\rho} u_t \Pi, \tag{1}$$

$$R_t = (1-\lambda)R_{t-1} + \frac{\tau_t}{P_t}, \tag{2}$$

where u_t are the innovations to uninformed investment and P_t is the share price. $\Pi \leq 1$ allows uninformed investment to diminish over time. In the case of Motors Liquidation, one source of uninformed investment is investors reacting to news about the reorganized General Motors. High values of τ_t may reflect otherwise rational investors responding to favorable news about GM, and seeking to buy its shares, but buying MTLQQ instead. In one egregious example, Motors Liquidation's share price surged from below fifty cents to eighty-three cents per share in early January 2010, apparently in response to the opening of an automotive show in Detroit.[7] The price spiked again in August 2010 after GM filed paperwork with the SEC in preparation for its initial public offering. Neither event did anything to resuscitate Motors Liquidation's moribund prospects.

If all agents are uninformed, then the price equals the ratio of uninformed investment to the number of available shares:

$$P_t^u = \frac{\tau_t}{N - (1-\lambda)R_{t-1}}. \tag{3}$$

After General Motors declared bankruptcy, few additional short sales of the stock occurred.[8] Furthermore, it is questionable whether uninformed investors are sufficiently sophisticated to engage in short sales even if they are allowed. Our second core assumption is therefore that rational agents may choose to enter the market only by purchasing shares at the equilibrium price, and that they therefore cannot profit when the share price is high by taking a short position. If all agents are rational and if the interest rate equals $\beta^{-1} - 1$, then the share price equals[9]

$$P_t^r = \beta E_t[P_{t+1}]. \tag{4}$$

Equation (4) represents a standard asset-pricing model, except that it omits dividend payments. It follows that, if all agents are rational, then the unique steady state price equals zero. There are, however, two potential share prices. If all agents are uninformed, then the equilibrium price is determined by (3). If $P_t^u < \beta E_t[P_{t+1}]$, however, then rational investors will enter the market. We assume that there are enough rational investors so that arbitrage will then cause $P_t = P_t^r$. At equilibrium,

$$P_t = \max\left[P_t^r, P_t^u\right]. \tag{5}$$

Equations (1)–(5) fully characterize the model.

We calibrate our model to Motors Liquidation. We set $N = 610,600,000$, matching the total shares of MTLQQ. We set $\lambda = 1.12\%$, so that our simulated daily volume equals MTLQQ's average daily volume. We set the discount factor, β, equal to 0.9999 to reflect our use of daily data. We set the steady state level of τ so that the simulated average share price matches the sample average for Motors Liquidation. We set $\Pi = 0.9983$, matching the daily estimated trend in the MTLQQ data.

Higher values of ρ, the AR(1) coefficient for $\ln(\tau_t)$, increase the observed persistence of the price in both states. Increasing the variance of u_t increases the observed variance of the share price, whereas the mean of u_t determines the average price. We calibrate these three parameters so that the simulated share price, conditional on its being above its average value, closely matches the actual values of MTLQQ's share price, conditional on its being above its mean. For simulations of the model, we impose that u_t is a uniform distribution between 1.110 and 1.356, and $\rho = 0.874$.[10]

The equilibrium share price exhibits two notable features. First, as long as $\tau_t > 0$ $\forall t$, the model's only equilibrium is a bubble equilibrium because the equilibrium price is always positive despite the asset having no fundamental value.[11] Second, higher values of τ_t increase the frequency of an uninformed share price. When there is a high level of uninformed investors, rational investors expect the share price of the asset to be bid down as τ_t converges downward to its steady state. They therefore exit the market. If τ_t is low, then rational investors expect that as τ_t converges upward to its steady state, the share price will increase as well. Rational investors therefore enter the market, which results in a higher price than if only

TABLE 1. Simulated AR(1) processes for different price types

Price type	α	ψ	$(\sigma_u^2)^i$	Frequency	Uninformed shares	Uninformed vol.
Rational–rational	0.049	0.935	0.00035	75.63%	99.38%	68.51%
Uninformed–uninformed	0.234	0.728	0.00223	19.21%	100.00%	100.00%
Uninformed–rational	0.634	0.159	0.00005	2.58%	99.98%	97.88%
Rational–uninformed	0.072	0.953	0.00092	2.58%	100.00%	100.00%

uninformed investors were to hold the stock. Rational agents thus act as a price floor.

The frequency of the rational state depends on the variance of u_t. If u_t is a constant, then there is no chance of the price increasing over time and rational agents never buy shares. As the variance of u_t increases, however, rational agents are more likely to buy shares in the hope that a large positive shock to u_t will cause a high share price in the next period.

The model includes an important nonlinearity. As the number of shares held by rational investors increases, the uninformed share price decreases dramatically. In this case, the model may remain in a stable and rational state for longer than suggested by a linearized model. We thus directly simulate the nonlinear model and fit the resulting price to a simple AR(1) process: $\hat{P}_t^i = \alpha + \psi \hat{P}_{t-1}^i + u_t^i$, where $\hat{P}_t = \frac{P_t}{\Pi^t}$, the detrended price.[12] For the following discussion we define the persistence as the observed AR(1) coefficient. Table 1 reports the results.[13]

Rational agents act as a price floor, which results in a highly persistent and stable equilibrium price. The uninformed state is therefore far more volatile than the rational state. The column “Uninformed shares” indicates the average fraction of total shares held by uninformed investors. The column “Uninformed vol.” indicates the average fraction of traded shares purchased by uninformed investors. The model is in the rational state most of the time. Even in the rational state, however, rational agents hold only a small fraction of the shares. In the rational state, uninformed agents are purchasing a well-priced asset and are thus unharmed by their mechanical style of investing. However, 22% of the time, the model is in the uninformed state, and rational investors exit the market. In this state, uninformed investors will likely experience a loss on their stock purchases. This result suggests that a small number of rational speculators were sufficient to make the MTLQQ share price act rationally 78% of the time.

The volume of uninformed investment in Motors Liquidation likely results from the long history of General Motors, the size and news coverage of its bankruptcy, and the fact that it sold such a widely used consumer good. Surely, however, stocks with significant fundamental values also attract some uninformed investors. If uninformed investment, however, is never sufficient without any rational agents to drive the price above its fundamental value, then the presence of uninformed

TABLE 2. Simulated AR(1) processes for different price types

Price type	α	ψ	$(\sigma_u^2)^i$	Frequency	Uninformed shares	Uninformed vol.
Low–low	0.102	0.858	0.00024	54.66%	99.71%	82.48%
High–high	0.205	0.759	0.00165	31.73%	99.93%	95.24%

agents will not affect the price. In this case, uninformed investors are unharmed by the mechanical nature of their investment. The relevance of our model is therefore limited to cases where uninformed investment is high relative to the stock's fundamental value. The most obvious such case is Motors Liquidation. To test our model, we now compare its predicted dynamics with the actual time series of Motors Liquidation's daily share price.

3. ESTIMATION

We use data on the daily closing price of MTLQQ. We begin our time series on 8/3/09, allowing one month to pass since Motors Liquidation's 7/1/09 statement confirming the company's grave prognosis. Our data continue through 9/30/10, allowing ample time between the end of our sample and GM's IPO. We fit the data to a linear trend, and detrend using $\hat{P}_t = P_t - 0.00131t$. The average daily detrended closing price is \$0.765.

The state of the model depends on the unobservable variables τ_t and R_{t-1}. We cannot therefore directly estimate the behavior of Motors Liquidation in each state. The state of the model, however, is correlated with the share price, and we exploit this relationship to test the predictions of our model indirectly. Specifically, the uninformed state generally occurs only when the share price is above its average. We begin by refitting the simulated model to AR(1) processes, depending on whether the simulated share price is below its average value. Table 2 reports the results.[14]

The rational state exhibits more persistence and less volatility than the uninformed state. Because low prices very likely correspond to the rational state, when the price is below average, it is very stable and persistent. High prices, however, correspond to both states, as well as the transitions between them. The high-price properties are thus a weighted average of all four rows of Table 1. As a result, when prices are high, they are also more volatile and less persistent.

After detrending, we repeat the exercise from Table 2 for the actual Motors Liquidation share price. Table 3 reports the results.

We have calibrated our model to match MTLQQ's data when prices are high. The estimates for when prices are low are left as a result of the model. A standard asset-pricing model where the share price depends only on fundamentals predicts a large discrepancy neither between $(\sigma_u^h)^2$ and $(\sigma_u^l)^2$, nor between ψ_l and ψ_h. Our model, however, does, and Table 3 shows that both features are clear in the data.

TABLE 3. Estimated AR(1) processes for different price types

Price type	α	ψ	σ_u^2
Low–low	0.127 (0.028)	0.823 (0.039)	0.00043
High–high	0.197 (0.060)	0.760 (0.073)	0.00166

TABLE 4. Estimated AR(1) processes for the Dow 30

Stock	ψ^l	ψ^h	$\psi^l - \psi^h$	$(\sigma_u^2)^l$	$(\sigma_u^2)^h$	$\frac{(\sigma_u^2)^h}{(\sigma_u^2)^l}$	Rank $(\psi_l - \psi_h)$	Rank (var. rat.)
MTLQQ	0.823	0.760	0.062	.00043	0.00166	3.86	2	1
CSCO	0.904	0.947	−0.043	0.066	0.104	1.57	15	2
HD	0.880	0.813	0.066	0.885	0.766	0.87	1	20
UTX	0.923	0.872	0.052	0.340	0.223	0.66	3	30
DJIA			−0.058			0.998		

These results are therefore evidence in support of our hypothesis that uninformed agents, and not fundamentals, are responsible for MTLQQ's surprisingly large market capitalization.

Because our model is simple, it is possible that other factors, ignored in our model, could be driving our results. To rule out this possibility, at least for the stock market as a whole, we repeat the same exercise from Table 3 for all 30 detrended components of the Dow Jones Industrial Average (DJIA) for the same time period.[15] Table 4 reports the results.[16]

The predictions of our theoretical model are limited to assets with very low fundamental values, none of which are currently in the DJIA. No other stock yields a variance ratio near that of either MTLQQ (3.86) or our fitted model (6.83). The next largest ratio is Cisco, at just 1.57. The average ratio across the 30 DJIA stocks is just below one. These results show that an important feature of MTLQQ's data is not systematic across all stocks. Although the result it less pronounced, the data also support our prediction about the AR(1) coefficients. Only one of the DJIA stocks (Home Depot) has a larger gap between the low-price AR(1) coefficient and its high-price counterpart. We assert that these results support our claim that MTLQQ is, uniquely among the 31 stocks analyzed, driven by uninformed investors who create a bubble equilibrium.

4. CONCLUSION

The financial press has speculated that irrational behavior fueled heavy trading in MTLQQ, and noted the presence of rational speculators. We reconcile these two seemingly disparate events by developing an asset-pricing model with both types of agents. We show that a surprisingly small number of rational agents are able to make the market as a whole rational most of the time. Less frequently, however, uninformed investors inflate the share price and drive rational agents from the market. Our model predicts that high share prices are far more volatile than low share prices. Empirical estimation supports this prediction for MTLQQ, but not the stocks that compose the Dow Jones Industrial Average.

NOTES

1. See Chris Isidore, "Dumbest Moments in Business 2009," *Fortune*; Paul R. La Monica, "Drive away from MTLQQ as fast as you can," *CNNMoney.com*, 1/19/10; Floyd Norris, "The Old G.M. Is Dead, but Its Shares Live On," *The New York Times*, 10/30/09; and Bernard Simon, "Old GM shares make unlikely resurgence," *The Financial Times*, 1/18/10.

2. One GM employee is quoted in *The Financial Times* as saying that "It's been a challenge to get through to people that these are not shares in the new company."

3. Other traders may include those collecting old GM stock as a "souvenir," and traders who committed to short sales prior to GM's bankruptcy. The Securities and Exchange Commission effectively prevented additional short sales after the bankruptcy, and the number of outstanding short sales has decreased dramatically without a corresponding decrease in the average share price. In our model, both of these groups act as uninformed investors.

4. See, for example, DeLong et al. (1990), Timmermann (1993, 1996), Brock and Hommes (1998), Chakraborty and Evans (2008), Granato et al. (2008), and Branch and Evans (2010).

5. See, for example, Olsen (1998), Hirshleifer (2001), and Shiller (2002).

6. λ may arise from uninformed agents selling shares in order to increase their liquidity, or from uninformed agents selling shares after realizing that they own MTLQQ and not GM stock.

7. See Paul R. La Monica, "Drive away from MTLQQ as fast as you can," *CNNMoney.com*.

8. SEC Rule 204, temporarily instituted in 2008 and permanently adopted in July 2009, clearly forbids "naked" short selling, or selling short without literally borrowing shares, a practice which in the past had been used to short bankrupt companies. Anecdotal evidence suggests that finding a lender for a short sale of MTLQQ is nearly impossible (*Seeking Alpha,* August 30, 2009). Moreover, the short interest in MTLQQ declined from 100M shares on June 15, 2009 to only 25M shares in October 2009, less than 4% of all shares, and only a few days worth of volume (*Traders Magazine,* November 2, 2009).

9. We also assume that if uninformed investment is eventually to collapse, then rational agents are able to correctly forecast the date of collapse and exit beforehand. Adding a probability of τ_t collapsing is identical to assuming a lower discount factor and does not affect our conclusions.

10. Simulations confirm that a normal distribution with the same mean and variance yields very similar results.

11. For this model, a bubble solution does not exist when all agents are rational. Other papers show, however, that bubble solutions that depend on agents' self-fulfilling expectations may exist in similar models under adaptive learning. See Evans and McGough (2005, 2011) and Gershun and Harrison (2008). Likewise, the rational bubbles literature obtains fragile conditions where bubble equilibria may occur even if all agents are fully rational. Unlike the present paper, that literature derives an equilibrium price that follows an explosive process. See Santos and Woodford (1997).

12. We set the simulation length to 100,000 periods. The standard errors for the fitted AR(1) processes are thus near zero.

13. "Rational–rational" refers to observations where the price remains rational for two consecutive periods, "Rationa–uninformed" refers to observations where the price switches from rational to uninformed, etc.

14. The low–low regression only takes the observations from the simulated model where the price was low for the current and previous period.

15. Along with Citigroup, GM was removed from the Dow Jones on June 8, 2009. The composition of the index has been unchanged since that time.

16. A handful of stocks appear to be $I(1)$. Differencing the data does not affect our conclusion. "Rank(var. rat.)" ranks the stocks by the ratio of the high-price to low-price volatility, and "Rank($\psi_l - \psi_h$)" ranks the stocks by the difference in AR(1) coefficients.

17. For additional discussion of nonlinearities in asset-pricing models, see Bidarkota (2006).

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APPENDIX: SIMULATING THE NONLINEAR MODEL

The model exhibits important nonlinearities, and we thus simulate the nonlinear model directly rather than relying on a linear approximation.[17] We begin by detrending and transforming τ_t, the innovations to uninformed investment. Denoting $\bar{u}$ as the steady state value of u_t, we define $\hat{\tau}_t = \frac{\tau_t}{\bar{u}^{1/(1-\rho)}} \Pi^t$. We then rewrite (1) as

$$\hat{\tau}_t = \hat{\tau}_{t-1}^{\rho} \hat{u}_t, \tag{A.1}$$

where $\hat{u}_t = u_t \bar{u}^{-1}$.

Prices are detrended in a similar way, so that $\hat{P}_t = P_t \Pi^{-t}$, $\hat{P}^u_t = P^u_t \Pi^{-t}$, and $\hat{P}^r_t = P^r_t \Pi^{-t}$. We also transform the total shares held by uninformed investors, R_t, into the fraction held by uninformed investors by dividing by N. Thus $\hat{R}_t = R_t/N$. We then rewrite (2) and (3) as

$$\hat{R}_t = (1-\lambda)\hat{R}_{t-1} + \frac{\bar{u}^{1/(1-\rho)}\hat{\tau}_t}{N\hat{P}_t}, \tag{A.2}$$

$$\hat{P}^u_t = \frac{\hat{\tau}_t \bar{u}^{1/(1-\rho)}}{N(1-(1-\lambda)\hat{R}_{t-1})}. \tag{A.3}$$

Finally, we rewrite (4) and (5) as

$$\hat{P}^r_t = \beta \Pi E_t[\hat{P}_{t+1}], \tag{A.4}$$

$$\hat{P}_t = \left[\hat{P}^r_t, \hat{P}^u_t\right]. \tag{A.5}$$

The only complication of simulating (A.1)–(A.5) is calculating $E_t[\hat{P}_{t+1}]$. We employ a simple iterative algorithm. We create a discrete approximate state space by dividing both the intervals (τ^-, τ^+) and (R^-, R^+) into g equal increments. The approximate state space thus contains $(g+1)^2$ couplets $(\hat{\tau}_k, \hat{R}_l)$. We then divide the interval (u^-, u^+) into d identical increments. The latter step creates a discrete distribution that approximates the true distribution of u_t.

For each of the $(g+1)^2$ couplets that constitute the approximate state space, we calculate the corresponding uninformed price using (A.3). We then impose the initial conditions by setting the rational price equal to the uninformed price for each couplet: $\hat{P}^r_0 = \hat{P}^u_0 \; \forall(\hat{\tau}_k, \hat{R}_l)$. Equations (A.2) and (A.5) then determine the equilibrium share price, $\hat{P}_0$, and the fraction of shares held by uninformed investors, $\hat{R}_0$.

We perform the same procedure for each couplet in the approximate state space, $(\hat{\tau}_k, \hat{R}_l)$. The jth iteration works as follows:

(i) We begin by setting $u_i = u^-$.
(ii) For $\hat{\tau}_k$, and this specific value of u_i, we calculate the one-period-ahead value of τ, which equals $\hat{\tau}^\rho_k \bar{u}^{-1} u_i$.
(iii) We find the couplet in the approximate state space that is closest to $(\hat{\tau}^\rho_k \bar{u}^{-1} u_i, R(\hat{\tau}_k, \hat{R}_l))$. We denote the corresponding values of the share price, computed in the previous iteration, as P_i.
(iv) We repeat steps (i)–(iii) for $u_i = u^- + \frac{1}{d}(u^+ - u^-), u_i = u^- + \frac{2}{d}(u^+ - u^-) \ldots u_i = u^+$.
(v) For a uniform distribution, the one-period-ahead expectation of the price is a simple average of the previous d P_i's. For a normal distribution, the average must be weighted by the pdf. The results are very similar for both distributions for a common mean and variance.
(vi) We then update the rational price according to

$$\hat{P}^r_j(\hat{\tau}_k, \hat{R}_l) = \iota\beta\Pi E_t[\hat{P}_{j+1}(\hat{\tau}_k, \hat{R}_l)] + (1-\iota)\hat{P}^r_{j-1}(\hat{\tau}_k, \hat{R}_l), \tag{A.6}$$

where ι is the gain. Higher gains imply more rapid updating.
(vii) We then compute the equilibrium price and fraction of shares held by uninformed investors using (A.2) and (A.5).

We continue the procedure until convergence has occurred. For the results in the paper, we set $g = 100$, $d = 50$, and $\kappa = 0.15$. We run 100 iterations and verify that convergence has occurred.

The approximate state space provides a mapping between $(\hat{\tau}_t, \hat{R}_{t-1})$ and the model's equilibrium. It is then straightforward to simulate the model. Starting the system at and $\hat{\tau}_t = 1$, we randomly generate 100,000 draws for u_t. Each period, we match $(\hat{\tau}_t, \hat{R}_{t-1})$ to the nearest couplet in the approximate state space. Finally, we verify that all simulated couplets $(\hat{\tau}_t, \hat{R}_{t-1})$ are inside the intervals (τ^-, τ^+) and (R^-, R^+).